Written Homework #4

① $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\2\end{bmatrix}$, $T\left(\begin{bmatrix}0\\2\end{bmatrix}\right)=\begin{bmatrix}-2\\2\\0\end{bmatrix}$, $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=\begin{bmatrix}-1\\2\\2\end{bmatrix}$

$\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\2\end{bmatrix} \Rightarrow$ $a+b=1$, $c+d=1$, $e+f=2$

$\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}-2\\2\\0\end{bmatrix} \Rightarrow$ $2b=-2 \Rightarrow b=-1$, $2d=2\Rightarrow d=1$, $2f=0\Rightarrow f=0$ $\Big\}$ $a=2$, $c=-1$, $e=2$

$\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}-1\\2\\2\end{bmatrix}$

$a+3b=-1 \Rightarrow (2)+3(-1)=-1$ ✓

$c+3d=2 \Rightarrow (-1)+3(1)=2$ ✓

$e+3f=2 \Rightarrow (2)+3(0)=2$ ✓

So:

(a) $\begin{bmatrix}2&-1\\-1&1\\2&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=T(x)$

(b) The answer would become impossible because the linear transformation in the third equation would imply $e+3f=1$, and from the first two equations $e+3f=2$. So this answer would be inconsistent and therefore $T(x)$ would not be a linear transformation.

② (a) T cannot be 1-1, because T is only 1-1 if and only if the only solution to $T(x)=0$ is $x=0$, and $x\neq 0$ in this situation.

(b) T can be onto, because $T(x)=0$ doesn't mean that there can't be an element in the domain that corresponds to each element in the codomain.

For example: $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix} \Rightarrow T\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)=\begin{bmatrix}1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

This is an example of a nonzero vector $\vec{x}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and an onto transformation matrix $\begin{bmatrix}1&1&1\\0&1&0\end{bmatrix}$ which creates an output = to 0.

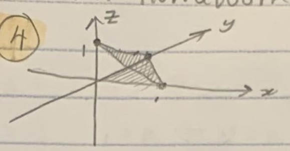
(c) $T(u)$ & $T(v)$ must be linearly dependent if u & v are linearly dependent because $T(v)=T(cu)$ where c is whatever scalar makes $v=u$. Then $T(v)=cT(u)$ and they are linear combinations of each other, thus making them linearly dependent.

(d) $T(u)$ & $T(v)$ are not guaranteed to be linearly independent because the transformation matrix $\begin{bmatrix}0&0&0&\cdots\\0&0&0&\cdots\\0&0&0&\cdots\\\vdots&\vdots&\vdots&\end{bmatrix}$ could turn any set of linearly independent vectors into the $\vec{0}$ vector, making them linearly dependent.

③ In order for T to be a linear transformation, $T(v+u)=T(u)+T(v)$ (additive property) and $T(cu)=cT(u)$. Therefore, $w=0$ because $T(u+v)=u+v+w=T(u)+T(v)$ (if $w=0$) and $T(cu)=c(u+w)=cu$ (if $w=0$) and $T(v+u)=(v+u)+w\neq v+u+2w$ (if $w\neq 0$) and $cT(v)=c(v+w)\neq cv+w$ (if $w\neq 0$)

Written Homework 4 (continued)

(4)



Going from $(x,y,z) \Rightarrow (x,y)$ removes the z-coordinate so the final corners are: $(1,0)$, $(0,1)$, $(0,0)$. This is a linear transformation because it preserves the additive property and scalar multiplication:

$$\left.\begin{aligned} &T((x_1,y_1,z_1)+(x_2,y_2,z_2)) = (x_1+x_2,\ y_1+y_2) \\ &T(c(x,y,z)) = (cx,\ cy) \end{aligned}\right\} \text{(a)}$$

The area of the image after projection is $\frac{1}{2}\left|1(1-0)+0(0-0)+0(0-1)\right|$

$= \frac{1}{2}$ (b)

(c) Yes, because a linear transformation can scale the original vectors, including preserving the original area by adding a scaling factor while mapping one of the corners to $(0,0)$

(5) $S \circ T$ cannot be 1-1 because any function that goes from $\mathbb{R}^3 \Rightarrow \mathbb{R}^2$ cannot have a unique output value mapped from each input value, and any output after that $(\mathbb{R}^2 \to \mathbb{R}^4)$ would not change that.

Written Homework 4

(1) (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\2\\0\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\2\end{bmatrix},$$

or if it's impossible, explain why.

(b) How does your answer change if the third condition changes to

$$T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\1\end{bmatrix}?$$

(2) Assume $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

(a) Suppose there is a nonzero vector $\mathbf{x} \in \mathbb{R}^{\mathbf{m}}$ such that $T(\mathbf{x}) = \mathbf{0}$. Is it possible that T is one-to-one? Give an example, or explain why it's not possible.

(b) Suppose there is a nonzero vector $\mathbf{x} \in \mathbb{R}^{\mathbf{m}}$ such that $T(\mathbf{x}) = \mathbf{0}$. Is it possible that T is onto? Give an example, or explain why it's not possible.

(c) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent vectors in $\mathbb{R}^m$. Show that $T(\mathbf{u})$ and $T(\mathbf{v})$ are also linearly dependent.

(d) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly *independent* vectors in $\mathbb{R}^m$. Is it guaranteed that $T(\mathbf{u})$ and $T(\mathbf{v})$ are also linearly independent? If yes, explain why. If no, give an example where this is not the case.

(3) Let $\mathbf{w} \in \mathbb{R}^n$ and suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T(\mathbf{v}) = \mathbf{v} + \mathbf{w}$. Determine the exact conditions on $\mathbf{w}$ that make T a linear transformation.
(First, show that if your condition on $\mathbf{w}$ is satisfied, then T is a linear transformation. Then show that if your condition on $\mathbf{w}$ is *not* satisfied, then T is *not* a linear transformation.)

(4) Consider the triangle Δ in $\mathbb{R}^3$ with corners $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

(a) Find the image of Δ under the projection that sends $(x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2$. Is this a linear transformation?

(b) What the area of the image of Δ after the projection?

(c) Is there a linear transformation that will send Δ to the xy-plane so that the image has the same area as Δ, and one of the corners of the image is $(0, 0)$?

(5) Say we have linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$. Let $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be the composition (that is, $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^4$).
Can $S \circ T$ be one-to-one?
(Hint: Start by thinking about all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.)